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Gravitational stability of compressible thermal fluid layer under magnetic viscosity

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Abstract : The gravitational stability of compressible thermal fluid layer under the effect of magnetic viscosity is investigated. The dispersion relation is derived and discussed. It is found that when the instability sets in as stationary convection, the compressibility has a stabilizing influence. It is found also that the magnetic viscosity effect is always stabilizing, for $\kappa < \nu$ and $\kappa < \eta$, the overstability cannot occur and the principle of the exchange of stability is valid.

Keywords : Gravitational stability, compressible thermal fluid, magnetic viscosity

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1. Introduction

The study of thermal convection in our environment is of importance in geophysics, oceanography, meteorology and many other practical fields and it is meaningful to have extensions in the fields of hydrodynamic and hydromagnetic stability.

The stability of a fluid heated from below is now a classical problem. The convection resulting from this instability was studied experimentally even much earlier by Benard [1].

The magnetic viscosity effect was first deduced by Marshall [2] and first used in this form in an analysis of the flute type gravitation instability of a magnetized plasma by Rothenbluth *et al* [3].

An early investigation of the influence of rotation on thermal convection was presented by Nakagawa and Frenzen [4]. The combined effects of a magnetic field and the Coriolis force were first studied by Chandrasekhar [5].

Chandrasekhar [6] has demonstrated, in his well known book, the thermal instability of a fluid layer heated from below under the influence of vertical magnetic field.

Radwan and Elazab [7] considered the magnetohydrodynamic gravitational stability of a homogeneous fluid medium with variable streaming velocity.

The object of the present work is to study the thermal instability of a gravitational compressible fluid layer with finite Larmor radius under the influence of a vertical magnetic field. Pradhan *et al* [8] considered a fluid layer in a variable gravitational field heated from below.

2. Formulation of the problem and perturbed state

Consider an infinite horizontal layer (thickness d), in which a uniform temperature gradient $\beta\,(= -\,dT/dz)$ is maintained. The layer is acted upon by the gravitating force $\boldsymbol{g} = (0, 0, -g)$ and a uniform vertical magnetic field $\boldsymbol{H} = (0, 0, H)$.

Since the set of equations governing the flow of compressible fluids is quite complicated, Spiegel and Veronis [9] have made, to simplify the set governing equations, the following assumptions :

(i) The depth of fluid layer is much smaller than the scale height as defined by them.

(ii) The fluctuation in temperature, pressure and density, introduced due to motion, has exceeded their static variations.

According to these assumptions, the temperature, pressure and density are given by

$$T(z) = -\beta z + T_o, \tag{1a}$$

$$p(z) = p_n - g\int_0^z (\rho_n + \rho_o)\,dz, \tag{1b}$$

$$\rho(z) = \rho_n\left[1 - \alpha_n(T - T_n) + \lambda_n(p - p_n)\right], \tag{1c}$$

where
$$\alpha_n = -\left(\rho^{-1}\frac{\partial\rho}{\partial T}\right)_n,\quad \lambda_n = \left(\rho^{-1}\frac{\partial\rho}{\partial p}\right)_n. \tag{1d}$$

p_n and ρ_n stand for the constant space distribution of pressure p and density ρ of the fluid while p_o and ρ_o stand for the temperature and density at the lower boundary $z = 0$.

The perturbation equations, under the above approximations, are given by

$$\partial \boldsymbol{u}/\partial t = -\rho_n^{-1}\nabla\delta P - g\alpha\theta + \nu\nabla^2\boldsymbol{u} + (\mu/4\pi\rho_n)(\nabla\times\boldsymbol{h})\times\boldsymbol{H}, \tag{2}$$

$$\nabla.\boldsymbol{u} = 0, \tag{3}$$

$$\partial\boldsymbol{h}/\partial t = \nabla\times(\boldsymbol{u}\times\boldsymbol{H}) + \eta\nabla^2\boldsymbol{h}, \tag{4}$$

$$\nabla.\boldsymbol{h} = 0, \tag{5}$$

$$\partial\theta/\partial t = (\beta - \frac{g}{c_p})w + \kappa\nabla^2\theta, \tag{6}$$

$$\alpha_n = T_n^{-1} = \alpha \text{ (say)},$$

$$\nu = \frac{\mu}{\rho_n}, \qquad \kappa = \frac{k}{\rho_n c_p},$$

where $\boldsymbol{u}$ (= (u, v, w)), $\boldsymbol{\theta}$, $\boldsymbol{h}$ and p denote respectively the perturbation in the velocity vector, temperature, magnetic field and stress tensor P. g/c_p, μ, v, k, κ and c_p stand for the adiabatic gradient, the dynamic viscosity, thermal conductivity, the thermal diffusivity, and the specific heat at constant pressure, respectively.

By using eq. (2), the Boussinesq Equation of State is

$$\delta\rho = -\alpha\rho_n\theta, \tag{7}$$

where α is the coefficient of thermal expansion.

According to Vandakurov [10], the stress tensor of the magnetic field can be written in the following form :

$$\begin{aligned}
P_{xx} &= p - \rho v_o\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ P_{yy} = p + \rho v_o\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\\
P_{zz} &= p,\ P_{xy} = P_{yx} = \rho v_o\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right),\\
P_{xz} &= P_{zx} = -2\rho v_o\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\ P_{yz} = P_{zy} = 2\rho v_o\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),
\end{aligned} \tag{8}$$

where $\rho v_o = NT/4\omega_B$, where ω_B is the ion gyration frequency, N and T refer to the number density and temperature of the ion in energy units.

3. Boundary conditions and eigenvalue relation

For analyzing the disturbances in terms of normal modes, the physical perturbation quantities can be put in the form

$$(w, \theta, \xi, h_z, \zeta) = (W(z), \theta(z), Z(z), Y(z), X(z)) \exp\left(i(k_x x + k_y y) + nt\right), \tag{9}$$

where k_x, k_y are the wave numbers along the x- and y-directions, $k^2 = k_x^2 + k_y^2$ is the resultant wave number and n is the growth rate. ξ and ζ denote, respectively, the vertical components of the circulation of $\boldsymbol{u}$ and $\boldsymbol{h}$ *i.e.*

$$\xi = ik_x v - ik_y u, \qquad \zeta = ik_x h_y - ik_y h_x.$$

Using eq. (9), eqs. (2) – (6) yield

$$(D^2-a^2)\,(D^2-a^2-\sigma)W = \frac{v_o d}{v}\,(2D^2+a^2)DZ + \left(\frac{g\alpha d^2}{v}\right)a^2\theta - \left(\frac{\mu H d}{4\pi\rho_n v}\right)(D^2-a^2)DY \tag{10}$$

$$(D^2-a^2-\sigma)Z = -\left(\frac{v_o}{v_d}\right)(2D^2+a^2)DW - \left(\frac{\mu HD}{4\pi\rho_n v}\right)DX \tag{11}$$

$$(D^2-a^2-r_1\sigma)X = -\left(\frac{Hd}{\eta}\right)DZ_1 \tag{12}$$

$$(D^2-a^2-r_1\sigma)Y = -\left(\frac{Hd}{\eta}\right)DW \tag{13}$$

$$(D^2-a^2-r_2\sigma)\theta = -\frac{\beta d^2}{\kappa}\left(\frac{K-1}{K}\right)W. \tag{14}$$

Here we have put

$$a = kd,\ \sigma = \frac{nd^2}{\gamma},\ r_1 = \frac{v}{\eta},\ r_2 = \frac{v}{\kappa},\ K = (c_p/g)$$

$$x/d = x',\ y/d = y',\ z/d = z' \text{ and } D = d/dz'.$$

Eliminating Z, X, $^{\mathrm{v}}$ and θ between eqs. (10) – (14) and simplifying, we get

$$\Big[(D^2-a^2)(D^2-a^2-\sigma^2)\{(D^2-a^2-\sigma)(D^2-a^2-r_1\sigma)-SD^2\}$$
$$+M^2(2D^2+a^2)^2 \times (D^2-a^2-r_1\sigma)\Big]D^2(D^2-a^2-r_1\sigma)(D^2-a^2-r_2\sigma)W$$
$$-S(D^2-a^2)\times\{(D^2-a^2-\sigma)(D^2-a^2-r_1\sigma)-SD^2\}(D^2-a^2-r_2\sigma)D^2W$$
$$+Ra^2\{(D^2-a^2-\sigma)(D^2-a^2-r_1\sigma)-SD^2\}(D^2-a^2-r_1\sigma)$$
$$\times((K-1)/K)W = 0 \tag{15}$$

where $S = \dfrac{\mu K^2 d^2}{4\pi\rho_n v\eta}$ is the Chandrasekhar number

and $M = v_o/v$ is a non-dimensional number including the effect of the magnetic viscosity.

and $R = \dfrac{g\alpha\beta d^4}{v\kappa}$ is the Rayleigh number.

Consider the case in which both the boundaries are free, and the medium adjoining the fluid is non-conducting. In this case, the boundary conditions are :

$$W = 0 \text{ and } \theta = 0 \text{ for } \xi' = 0 \text{ and } 1 \tag{16}$$

for, the surfaces $z' = 0$ and 1 are contained at constant temperatures and as such they can suffer no change; and it is also clear that the normal component of the velocity must vanish on these surfaces.

Since there is no tangential stress acting on the free surfaces and since the scalar pressure has no transverse components, this is equivalent to the vanishing of the components p_{xz} and p_{yz} of the viscous stress tensor

$$p_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

and
$$p_{yz} = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right). \tag{17}$$

Since w vanishes (for all x and y) on the bounding surface, it follows from eq. (16) that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \text{ on a free surface.} \tag{18}$$

From the equation of continuity (3) differentiated with respect to z, and using eqs. (9) and (18), we conclude that

$$D^2 W = 0. \tag{19}$$

The boundary conditions on the normal component of the vorticity can be deduced from the foregoing. Since

$$\xi = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{20}$$

it follows from eqs. (9), (16) and (17) that

$$D\,Z = 0. \tag{21}$$

Using eqs. (16), (19) and (21), one can show that all the even derivatives of W must vanish for $z = 0$ and 1 and, hence, the proper solution of eq. (15) characterizing the lowest mode is

$$W = W_o \sin \pi z, \tag{22}$$

where W_o is a constant.

Inserting eq. (22) into eq. (15) and introducing

$$q = a^2/\pi^2, \; R_1 = R/\pi^4, \; S_1 = S/\pi^2, \; i\sigma_1 = \sigma/\pi^2,$$

we obtain the following eigenvalue relation

$$R_1 = \left(\frac{K}{K-1}\right)\left[\frac{\{(1+q)\,(1+q+i\sigma_1)\,(1+q+ir_1\sigma_1) + S_1\,(1+q)\}\,(1+q+ir_2\sigma_1)}{q(1+q+ir_1\sigma_1)}\right.$$

$$\left. + \frac{M_1^2\,(q-2)^2\,(1+q+ir_2\sigma_1)\,(1+q+ir_1\sigma_1)}{q\{(1+q+i\sigma_1)\,(1+q+ir_1\sigma_1) + S_1)\}}\right]. \tag{23}$$

4. Stability discussions

To study the stability and instability cases as stationary convection if we put $\sigma = 0$, the dispersion relation tends to

$$R_1 = \left(\frac{K}{K-1}\right)\left\{(1+q)((1+q)^2 + S_1)/q + \frac{M^2(q-2)^2\,(1+q)^2}{q((1+q)^2 + S_1)}\right\}. \tag{24}$$

For fixed values of N, let the nondimensional parameter K (for compressibility effect) be kept as fixed. Then we find that

$$R_c = \left(\frac{K}{K-1}\right)^{-1} R'_c, \tag{25}$$

where R_c and R'_c denote the critical Rayleigh numbers in the absence and presence of compressibility, respectively. Thus, the effect of compressibility is to decrease the onset of thermal instability. Hence, the compressibility has a stabilizing influence. It is easy to show [from eq. (23)] that dR_1/dM is always positive for $K > 1$ and for $K < 1$, it is negative. This shows that the finite Larmor radius has a stabilizing effect. Thus, we conclude that the compressibility and magnetic viscosity effects are stabilizing the thermal instability.

Now, we shall study the possibility of instability setting in as overstability. For this, we need to determine the critical Rayleigh number for the onset of instability *via* a state of pure oscillations. Equating the real and the imaginary parts of eq. (25) and eliminating R we get

$$A\,\sigma_1^6 + B\,\sigma_1^4 + C\,\sigma_1^2 + D = 0, \tag{26}$$

where

$$A = r_1^4\,(1+q)\,(1+r_2), \tag{27}$$

$$B = r_1^2\,(1+q)^3\,(2+r_1^2+2r_2+r_2r_1^2) + r_1^2\,(1+q)\,(r_2-3r_1-2r_1r_2)S_1 + r_1^4\,R\,(r_2-1), \tag{28}$$

$$C = (1+q)^5\,(1+r_2+2r_1^2+2r_2r_1^2) + (1+q)^3\,\{(r_2-r_1)\,(1+r_1^2) + 2r_1^2(1+r_2) - 2r_1(r_2+1)\}\,S_1 + r_1^2S_1R^2(r_1+r_2) + (1+q)r_1(r_1r_2+3r_1-2r_2)\,S_1^2 + 2r_1^2\,(1+q)^2R^2(r_2-1), \tag{29}$$

$$D = (1+q)\,(S_1+(1+q))^2\left\{(1+q)^2(1+r_2) + S_1(r_2-r_1)\right\} + (1+q)^2$$
$$R^2\left\{(r_2-1)\,(1+q)^2 + (r_2+r_1)\,S_1\right\}. \tag{30}$$

Since, for overstability, σ_1 is real, the product of the roots of eq. (26) is $-D/A$ and if this is to be positive, then $D < 0$, since from (29) $A > 1$ and this is impossible, hence D is always positive if

$$r_2 > 1 \quad \text{and} \quad r_2 > r_1$$

which gives

$$\kappa < v \quad \text{and} \quad \kappa < \eta,$$

the sufficient conditions for the non-existence of overstability and the principle of the exchange of stability is valid.

5. Conclusions

The object of this analysis was to explore the stability of a gravitational compressible thermal fluid under magnetic viscosity. The stability criterion of the present problem is some what more general. Several reported works are recovered as limiting cases (see Chandrasekhar [6] and Pradhan *et al* [8]). The discussions of the present general stability criterion reveal the following results :

(i) When the instability sets in as stationary convection, the compressibility has a stabilizing effect. It is found also that the magnetic viscosity is always stabilizing.

(ii) For $\kappa < v$ and $\kappa < \eta$, it is found that overstability cannot occur and the principle of the exchange of stability is valid.

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